Fixed Point Objects Corresponding to Freyd Algebras

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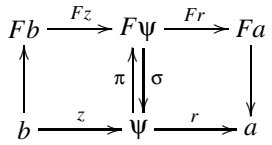
Abstract

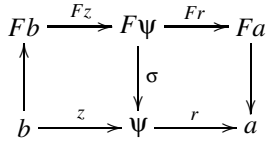
Fixed Point Objects and Freyd algebras are closely related. We consider a notion of Fixed Point Algebra that includes both Freyd algebras and Fixed Point Objects. We examine the transposition of such algebras in the context of an arbitrary adjunction rather than a strong monad. When the adjunction factors through certain comonads we obtain an account of parameterized fixed points. A simple lemma about morphisms from coalgebras to algebras explains one half of the transposition very cleanly. The other half requires a treatment of 'strict' maps and 'pointed' objects. At the risk of uncategorical results, we start with arbitrary classes of maps and objects and work out detailed conditions for the desired transposition result. Fixed Point Objects corresponding to Freyd algebras have unique fixed points in a strong sense. This allows for a treatment of fixed points based on generalized elements.

1 Introduction

A striking feature of Freyd's strengthened notion of free algebra [4] is its symmetry. A *Freyd invariant* is an initial algebra σ whose inverse π is a final coalgebra.

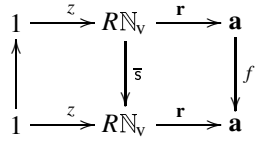


This definition is self-dual and interacts extremely well with functors on self-dual categories of the form $D^{\mathrm{op}} \times D$. However, because we are studying the transposition of σ to other categories, we adopt an equivalent strengthening that leaves π out of the picture. A *Freyd algebra* is an initial algebra σ that is final with respect to the obvious notion of morphism from coalgebra to algebra.



We want to understand the transposition of Freyd algebras because the categories that have Freyd algebras are not the categories in which the induced maps z and r are most useful. Suppose Freyd invariants are used to interpret recursive types and recursive types are used to type fixed point operators. Our prime example of a category with Freyd algebras is $D = \mathrm{cppo}_{\perp}$, the category of strict continuous functions on directed complete partial orders with least element, but fixed point operators in D are not very useful (except for finding least elements). A better semantic category is $C = \mathrm{cpo}$, the category of continuous functions on directed complete partial orders. Not only are fixed point operators in C worthwhile semantically, but the least fixed point operator is induced by the transpose of a Freyd algebra.

The categories D and C are related by an adjunction $L \dashv R$, where $L : C \to D$ makes continuous functions strict by adding least elements and $R : D \to C$ forgets strictness. The vertical natural numbers $\mathbb{N}_{\mathrm{v}} = \{0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq \ldots \sqsubseteq z\}$ can be made a Freyd algebra for the endofunctor LR on D. The structure map $\mathsf{s} : LR\mathbb{N}_{\mathrm{v}} \to \mathbb{N}_{\mathrm{v}}$ takes the new least element of $LR\mathbb{N}_{\mathrm{v}}$ to 0, each natural number to its successor and z to z. Its transpose $\overline{\mathsf{s}} : R\mathbb{N}_{\mathrm{v}} \to R\mathbb{N}_{\mathrm{v}}$ takes each natural number to its successor and z to z. This endofunction $\overline{\mathsf{s}}$ has special properties. It has a unique fixed point and is initial in the category of continuous endofunctions f on domains $\mathbf{a}$ with least element and strict continuous endofunction morphisms $\mathbf{r}$.



One checks that $r \circ z$ is the least fixed point of f.

We study this transposition of Freyd algebras into Fixed Point Objects axiomatically. From an axiomatic point of view, $\overline{\mathsf{s}}$ has special properties *because it is the transpose of a Freyd algebra*. These properties are not symmetric like the properties of s, but they are very useful semantically. They are equivalent to having a Fixed Point Object (for the monad RL) as described in [1] and [5]. Axiomatically, *this is why $r \circ z$ is the least fixed point of f*: given any adjunction $L \dashv R$ with R injective on objects, a Freyd algebra for LR induces a unique uniform family of fixed points in C. The family is uniform because of the factorization $r \circ z$. This corresponds to a symmetrical factorization in D but, due to the asymmetry of transposition, the mathematics of the correspondence is completely asymmetrical and so we organize our results into two distinct halves.

The first half of the paper, Sections 2, 3, 4 and 5, is limited to the finality condition corresponding to the final coalgebra part of a Freyd invariant and to the unique fixed point of a Fixed Point Object. A mathematical novelty is Lemma 1, but the main result is Lemma 2 which shows that Freyd algebras provide very robust unique fixed points, both plain and parametrized. The second half, Sections 6, 7 and 8, extends the results of the first half to the initiality condition corresponding to the initial algebra parts of a Freyd invariant and of a Fixed Point Object. The main result is Theorem 1 which uses the somewhat *ad hoc* notion of a Fixed Point Algebra to give a unified treatment of the transposition Freyd algebras to Fixed Point Objects and to parametrized Fixed

Point Objects. This gives an axiomatic framework for Freyd's observation that algebraic compactness induces a fixed point operator satisfying Plotkin's Axiom [3]: if D is algebraically compact, it has a Freyd algebra for LR which transposes to a Fixed Point Object in C which induces the unique fixed point operator satisfying Plotkin's Axiom. Moreover, by extending the adjunction $L \dashv R$ with the Kleisli adjunction for the comonad $c \times (\cdot)$ the same abstract argument shows that a Freyd algebra for $L(c \times R(\cdot))$ transposes to a parametrized Fixed Point Object which induces the unique parametrised fixed point operator satisfying a parametrized version of Plotkin's Axiom.

The first half of the paper is categorically satisfying and may even find applications beyond axiomatic domain theory. The second half of the paper is less satisfactory with its less than categorical treatment of 'strict' maps and 'pointed' objects. Our detailed inspection of the conditions on 'strict' maps and 'pointed' objects required for Theorem 1 suggests that less is needed than the usual theory suggests, but we have done little to organize the conditions categorically.

Contents and Terminology. In Section 2 we demonstrate two simple properties of morphisms from coalgebras to algebras. The associated diagram, with three sides equal to one, appears—or rather, due to a typo, does not appear—in Osius's categorical treatment of set theoretic recursion [6, Theorem 4.1] and in Taylor's categorical treatment of recursion in general [7, Definition 6.3.8]. We therefore use the term 'recursive morphism' in place of the term 'coalgebra-to-algebra morphism' used in [2]. This sits well with the equation $f = s \circ Ff \circ p$ which says that f can be rewritten in terms of itself, even if here we are concerned only with the notion of recursive morphism as a form of fixed point generalizing the equation $f = s \circ f$. In Section 3 we consider finality and initiality conditions with respect to recursive morphisms, in which case finality applies only to algebras and initiality to coalgebras. We call the initial coalgebras 'recursive' because they correspond to the domains of definition of recursive maps in [6] and [7]. Dually, we call the final algebras 'corecursive'. Using the properties demonstrated in Section 2, we observe that the formula for the transpose of a corecursive algebra $s : LSRa \to a$ produces another corecursive algebra $\overline{s} : SRa \to Ra$. A corecursive algebra for the identity endofunctor has a very robust form of unique fixed point. In Section 4 we analyse a strong notion of unique fixed point based on generalized elements which allows one to work without a specified pointing object. When an algebra is final for the endofunctor of a comonad, it has a form of unique parametrised fixed point. In Section 6 we introduce a notion of Fixed Point Algebra parametrized by both an endofunctor and arbitrary classes of 'strict' maps and 'pointed' objects. With added conditions on L and R, we show that the transpose of a Fixed Point Algebra $s : LSRa \to a$ produces another Fixed Point Algebra $\overline{s} : SRa \to Ra$. In Section 7 we consider the special case of Fixed Point Algebras for the identity endofunctor which we call Fixed Point Objects. If our setting is specialized to the monadic setting of [1] and [5], the existence of a Fixed Point Object in our sense is equivalent to the existence of a Fixed Point Object in their sense. We check that our definition induces the unique family of fixed points satisfying Plotkin's Axiom (interpreted with respect to the chosen classes of 'strict' maps and 'pointed' objects). In Section 8 we consider the special case of Fixed Point Algebras with all maps and objects 'strict' and 'pointed'. These are

equivalent to Freyd algebras. Putting everything together, we check that Freyd algebras induce unique families of fixed points and unique parametrized families of fixed points.

2 Recursive Morphisms

Suppose we have an endofunctor F. We write Coa$F$ and Alg$F$ for the usual categories of coalgebras and algebras for F.

Definition 1 *Given a coalgebra p for F and an algebra s for F, a map f gives a* recursive morphism *from p to s if $f = s \circ Ff \circ p$.*

$$p \overset{f}{\dashrightarrow} s$$

$$\begin{array}{ccc} Fb & \xrightarrow{Ff} & Fa \\ {\scriptstyle p}\uparrow & & \downarrow{\scriptstyle s} \\ b & \xrightarrow{f} & a \end{array}$$

Examples include parametrized fixed points, ordinary fixed points and recursive functions.

$$\begin{array}{ccc} N \times N & \xrightarrow{N\times 0} & N \times N \\ {\scriptstyle (\mathrm{id},\mathrm{id})}\uparrow & & \downarrow{\scriptstyle \mathrm{plus}} \\ N & \xrightarrow{0} & N \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{0} & N \\ {\scriptstyle \mathrm{id}}\uparrow & & \downarrow{\scriptstyle \mathrm{double}} \\ 1 & \xrightarrow{0} & N \end{array} \qquad \begin{array}{ccc} 1+N & \xrightarrow{1+\mathrm{double}} & 1+N \\ {\scriptstyle \mathrm{pred}}\uparrow & & \downarrow{\scriptstyle \langle 0,\mathrm{suc}\circ\mathrm{suc}\rangle} \\ N & \xrightarrow{\mathrm{double}} & N \end{array}$$

Suppose we have a natural transformation $\alpha : E \Rightarrow F$ from one endofunctor to another. Composition with α gives functors $\alpha \circ (\cdot) : \mathrm{Coa}E \to \mathrm{Coa}F$ and $(\cdot) \circ \alpha : \mathrm{Alg}F \to \mathrm{Alg}E$.

Proposition 1 *Given a coalgebra p for E and an algebra s for F, f gives a morphism from $\alpha \circ p$ to s if and only if f gives a morphism from p to $s \circ \alpha$.*

$$\frac{\alpha \circ p \dashrightarrow s}{p \dashrightarrow s \circ \alpha}\ .$$

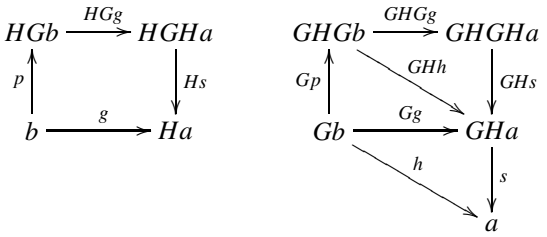
Proof. By the naturality of α we have $f = s \circ (Ff \circ \alpha) \circ p$ iff $f = s \circ (\alpha \circ Ef) \circ p$. □

Now suppose we have functors $G : B \to A$ and $H : A \to B$ (not necessarily adjoint). Then H carries algebras for GH to algebras for HG and, symmetrically and dually, G carries coalgebras for HG to coalgebras for GH. With respect to recursive morphisms, we obtain something of an adjunction.

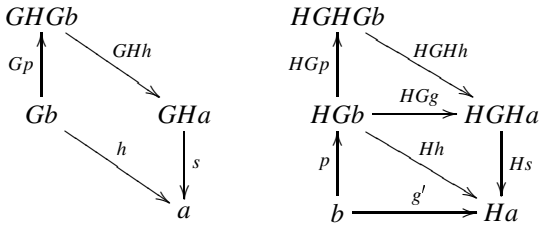
Lemma 1 *Given a coalgebra for p for HG and an algebra s for GH, there is a bijection, natural in the choice of p and s, between morphisms from Gp to s and morphisms from p to Hs.*

$$\frac{Gp \dashrightarrow s}{p \dashrightarrow Hs}$$

Proof. The bijection is given by $g \mapsto s \circ Gg$ and $h \mapsto Hh \circ p$. If g is a morphism from p to Hs, we see that $h = s \circ Gg$ is a morphism from Gp to s by applying G to the diagram for g.



Likewise by applying H to the diagram for h, we see that $g' = Hh \circ p$ is a morphism from p to Hs.



But then

$$g' = Hh \circ p = H(s \circ Gg) \circ p = Hs \circ HGg \circ p = g,$$

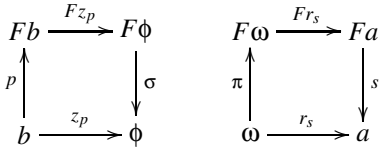
the morphism we started with. Similarly, $h' = s \circ G(Hh \circ p) = h$ starting with any morphism h. □

Given a natural transformation $\eta : \mathrm{Id} \Rightarrow RL$, we call $\overline{s} = Rs \circ \eta$ the *transpose* of s even if η is not the unit of an adjunction. Putting Lemma 1 together with Proposition 1, we obtain a correspondence between certain recursive morphisms into an algebra s for LR and arbitrary recursive morphisms into its transpose.

$$\frac{L(\eta \circ p) \dashrightarrow s}{p \dashrightarrow Rs \circ \eta}$$

3 Corecursive Algebras and Recursive Coalgebras

Definition 2 *An algebra σ is* corecursive *if for every coalgebra p there exists a unique morphism z_p from p to σ. Dually, a coalgebra π is* recursive *if there is a unique r_s from π to every algebra s.*

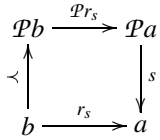


Proposition 2 (after Lambek) *The following are equivalent:*

1. *p is a final coalgebra.*
2. *p is the inverse of a corecursive algebra.*

However, not every corecursive algebra has an inverse. Also, unlike final coalgebras, corecursive algebras are not unique up to isomorphism. If $n \mapsto n-1$ is taken as a functor on the partial order $(\mathbb{Z}, \leq)$, then every algebra is corecursive (there are no coalgebras), no algebra has an inverse and no two are isomorphic.

If t is a terminal object, then the unique algebra $Ft \to t$ is a corecursive algebra. Dually, if i is an initial object, then the unique coalgebra $i \to Fi$ is a recursive coalgebra. By the General Recursion Theorem, every well-founded binary relation $\prec$ corresponds to a recursive coalgebra for the covariant power-set functor $\mathcal{P}$ on Set, but none of these is the inverse of an initial algebra for $\mathcal{P}$. For a categorical account of well-foundedness and the General Recursion Theorem, see Taylor [7, Section 6.3].



We apply Proposition 1 and Lemma 1 to the transport of corecursive algebras.

Corollary 1 *If σ is a corecursive algebra for F and α is a natural transformation from E to F, then $\sigma \circ \alpha$ is a corecursive algebra for E.*

Corollary 2 *If σ is a corecursive algebra for an endofunctor of the form GH, then $H\sigma$ is a corecursive algebra for HG.*

Given a natural transformation $\eta : \mathrm{Id} \to RL$ (not necessarily the unit of an adjunction), we can put these Corollaries together.

Corollary 3 *If σ is a corecursive algebra for LR, then its transpose, $R\sigma \circ \eta$, is a corecursive algebra for the identity endofunctor.*

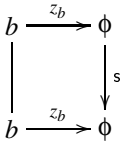
If we also have a natural transformation $\iota : S \to \mathrm{Id}$ (not necessarily the counit of a comonad), then a further application of Corollary 1 gives a corecursive algebra for S. Alternatively, we could start with a corecursive algebra τ for LSR, in which case the transformation ι is superfluous: by Corollary 2 with R for H and LS for G, $R\tau$ is a corecursive algebra for RLS and by Corollary 1 with $\eta_S : S \Rightarrow RLS$ for α, the transpose $\overline{\tau} = R\tau circ\eta_S$ is a corecursive algebra for S. This is so useful we make it a lemma and view Corollary 3 as the special case $S = \mathrm{Id}$.

Lemma 2 *Given functors $R : D \to C$ and $L : C \to D$, a natural transformation $\eta : \mathrm{Id} \to RL$ and an endofunctor $S : C \to C$, corecursive algebras for LSR transpose to corecursive algebras for S.*

Remark. The dual of Lemma 2 applies to recursive coalgebras: given a natural transformation $\varepsilon : FU \to \mathrm{Id}$ recursive coalgebras for UTF transpose to recursive coalgebras for T. We have in mind the factorization of a power set functor into an adjunction $F \dashv U$ topped with a monad T, in which case much more is true about recursive coalgebras.

4 Unique Fixed Points in General

An algebra for the identity endofunctor is an endomap. Suppose s is a corecursive algebra for the identity endofunctor, a *corecursive endomap*. Then, if by point we mean a map from some fixed object b, the endomap s has a unique fixed point z_b. Typically b is a terminal object 1 or a monoidal unit I, but because z_b exists uniquely not just for some b but for *all* b, our endomap has a unique fixed point in a strong sense.



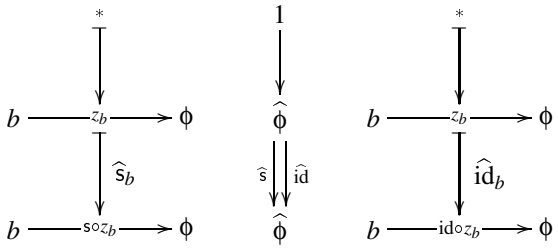
A *b-point in* ϕ is just any map from b to ϕ and a *fixed b-point for* $\mathsf{s} : \phi \to \phi$ is a b-point z such that $\mathsf{s} \circ z = z$.

Definition 3 *An endomap* s has a unique fixed point *if it has a unique fixed b-point* z_b *for each b.*

Proposition 3 *Corecursive endomaps have unique fixed points.*

Corollary 4 *Given a natural transformation* $\eta : \mathrm{Id} \Rightarrow RL$*, corecursive algebras for LR transpose to endomaps with unique fixed points.*

In Set, s has a unique fixed point when there is a unique element $z \in \phi$ such that $\mathsf{s}(z) = z$. In a general category C, the maps z_b behave as maps sending everything to one particular point, a sort of virtual element z. In terms of the Yoneda representation of C, which represents a map f as the transformation $\widehat{f}$ taking z to $f \circ z$ in the category $[C^{\mathrm{op}}, \mathrm{Set}]$ of presheaves on C, b-points in ϕ are elements of $\widehat{\phi}$ at b and s has a unique fixed point when the transformation $\widehat{\mathsf{s}}$ fixes exactly one element of $\widehat{\phi}$ at each b, which is to say the equalizer of $\widehat{\mathrm{id}}$ and $\widehat{\mathsf{s}}$ is isomorphic to the constant presheaf 1.



More generally, a diagram d in C *has a unique fixed point* if $\lim \widehat{\mathrm{d}} \cong 1$ in $[C^{\mathrm{op}}, \mathrm{Set}]$. When the diagram consists of an endomap $\mathsf{s} : \phi \to \phi$ in parallel with the identity on ϕ we recover Definition 3. If C has a generating object, such a one element set in Set, our general notion of unique fixed point simplifies. If b is a generating object, then

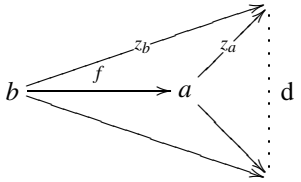
d has a unique fixed point *iff* there is a unique cone on d with vertex b.

Another useful characterization involves the category $\mathrm{Cone}(\mathrm{d})$ of cones and cone morphisms over the diagram d.

Lemma 3 *The following are equivalent:*

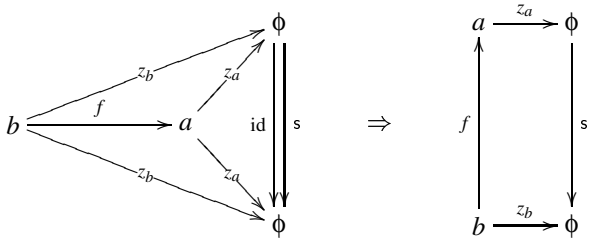
1. d *has a unique fixed point.*
2. *The vertex projection from* $\mathrm{Cone}(\mathrm{d})$ *to* C *is an isomorphism.*

When d has a unique fixed point, the inverse of the vertex projection is $b \mapsto z_b$ and every map $f : b \to a$ is a morphism from the cone z_b to the cone z_a



or, from the point of view of the cones, precomposition with an arbitrary map f from b to a carries the cone z_a to the cone z_b. When f is an endomap, precomposition with f has no effect at all.

Corollary 5 *If an endomap* s *has a unique fixed point, then for any* $f : b \to a$ *we have* $z_b = \mathsf{s} \circ z_a \circ f$.



In the case of a corecursive endomap, this means the maps z_p depend solely on the carrier of p. It does not mean an endomap with a unique fixed point is a corecursive endomap. The notion of corecursive endomap is stronger because $\mathsf{s} \circ z \circ p = z$ may hold for z other than z_b.

Counterexample 1 If $\mathsf{s} : 3 \to 3$ fixes one element and swaps the other two, then $\mathsf{s} \circ \mathsf{s} = \mathrm{id}$. This gives $\mathsf{s} \circ \mathrm{id} \circ \mathsf{s} = \mathrm{id}$ and $\mathsf{s} \circ \mathsf{s} \circ \mathsf{s} = \mathsf{s}$ and so, if s is taken for p, both id and s can be taken for z_p, but $\mathsf{s} \neq \mathrm{id}$.

More generally, if s is a corecursive endomap and there exists s^- such that $\mathsf{s} \circ \mathsf{s}^- = \mathrm{id}$, then s is an identity and hence the identity on a terminal object. In particular, if s is a corecursive automorphism, then it is the identity on a terminal object.

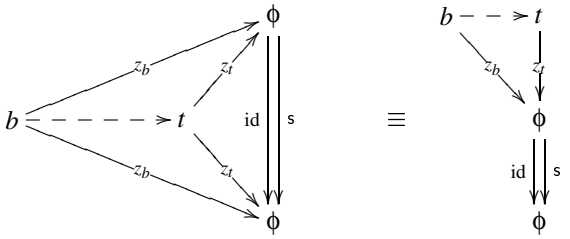
Limits. The relationships between limits, terminal objects and unique fixed points can get a little confusing. The Yoneda representation preserves limits, so if d has limit t and t is terminal, then $\widehat{\mathsf{d}}$ has limit $\widehat{t}$ and $\widehat{t}$ is terminal which means d has a unique fixed point. Converses follow from Lemma 3. When d has a unique fixed point,

$$t \text{ is terminal } \textit{iff } z_t \text{ is a limit cone}$$

or, looked at another way, when C has a terminal object t,

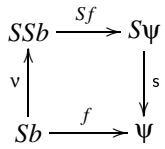
$$\mathsf{d} \text{ has a unique fixed point } \textit{iff } \mathsf{d} \text{ has limit } t \text{ in } C.$$

In particular, in the presence of an endomap s with a unique fixed point, an object t is terminal *iff* z_t equalizes id_ϕ and s or, looked at the second way, in the presence of a terminal object t, an endomap s has a unique fixed point *iff* t equalizes id_ϕ and s.



5 Parametrized Fixed Points

Suppose $S : C \to C$ carries comonad structure with comultiplication $\nu : S \Rightarrow SS$ and counit $\iota : S \Rightarrow \mathrm{Id}$. The diagram in C that says f is a morphism from ν_b to s is the same diagram that says f gives a fixed b-point for the arrow given by s in the Kleisli category C_S.



Proposition 4 *Taken as an endomap in the Kleisli category for S, a corecursive algebra for S has a unique fixed point.*

While this follows directly from the observation above, it can also be seen as Corollary 4 with the left and right Kleisli adjoints, K_L and K_R, for L and R. In this adjunction, the transpose of a map $s : Sa \to a'$ in C is the map from a to a' in C_S given by s. Note that we are not using the fact that K_L is left adjoint to K_R: the comonad structure is

used to build the natural transformation $\mathrm{Id}_{C_S} \Rightarrow K_R K_L$, which is all we need. Note also that corecursive algebras for S actually transpose to corecursive endomaps, which is stronger and harder to see directly. Here is another instance of Corollary 4.

Corollary 6 *Given functors R and L, a natural transformation $\eta : \mathrm{Id} \to RL$ and a comonad S, corecursive algebras for LSR transpose to give endomaps in C_S with unique fixed points.*

This can be seen either as Lemma 2 followed by Proposition 4 or as Corollary 4 with $L \circ K_L$ and $K_R \circ R$ for L and R. Again, we actually obtain a corecursive endomap in C_S.

$$D \underset{L}{\overset{R}{\rightleftarrows}} C \underset{K_L}{\overset{K_R}{\rightleftarrows}} C_S$$

As we pointed out after Corollary 3, the transformation ι can also be used to obtain corecursive algebras for S from corecursive endomaps in C. When ι is the counit of comonad structure on S, this amounts to using K_R to carry endomaps in C to endomaps in C_S. Using $\eta : \mathrm{Id} \to RL$, corecursive endomaps in C can be obtained from corecursive algebras for LR, and so we have two ways of obtaining endomaps with unique fixed points in C_S, starting either with a corecursive algebra for LSR or one for LR. In applications, the second is seen to be a degenerate form of the first, with weaker properties.

Comonoids. The comonads we have in mind are given by multiplication, multiplication by any object in C if C has products or multiplication by a comonoid object if C is monoidal. As the first is a special case of the second, suppose C carries monoidal structure—a choice of products being one example—and c carries comonoid structure given by the maps

$$\delta : c \to c \otimes c \quad \text{and} \quad \kappa : c \to I$$

—which will be the diagonal and terminator for c when $\otimes$ gives products. The functor $c \otimes (\cdot)$ carries comonad structure with ν and ι given by the composite natural transformations

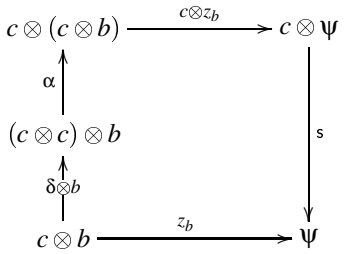
$$\alpha \circ (\delta \otimes (\cdot)) : c \otimes (\cdot) \Rightarrow c \otimes (c \otimes (\cdot)) \quad \text{and} \quad \lambda \circ (\kappa \otimes (\cdot)) : c \otimes (\cdot) \Rightarrow (\cdot),$$

where α and λ are the associativity and left identity natural transformations for the monoidal structure on C. We think of the Kleisli category C_c for this comonad as a category of c-parametrized maps: arrows in C_c are given by maps $c \otimes a \to a'$ in C.

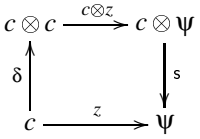
Definition 4 *A c-parametrized endomap $\mathsf{s} : c \otimes \psi \to \psi$* has a c-parametrized unique fixed point *if the endomap it gives in C_c has a unique fixed point in C_c.*

This means for each object b, there is a unique fixed b-point in the category C_c for the endomap given by s, which is to say, a unique c-parametrized map $z_b : c \otimes b \to a$ such that $f \circ (c \otimes z_b) \circ \nu_b = x$, using the definition of composition in C_c. Expanding ν in

terms of the monoidal structure, this becomes $f \circ (c \otimes z_b) \circ \alpha \circ (\delta \otimes b) = z_b$.



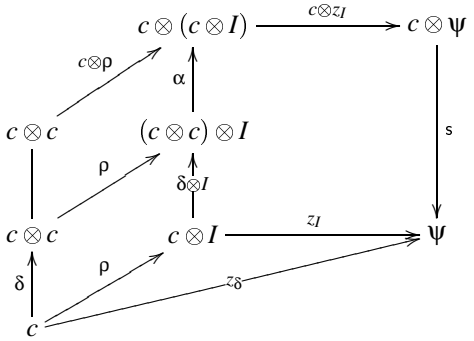
While Definition 4 is weaker than the notion of corecursive endomap in C_c, it is stronger than a naive definition based on the example of products in Set. If we define a 'c-parametrized fixed point' for s to be a map $z : c \to \psi$ such that $s \circ (c \otimes x) \circ \delta = x$



and ask for a unique such z, we are implicitly fixing $b = I$ where Definition 4 quantifies over arbitrary b. This can be seen in the proof below

Proposition 5 *A c-parametrized endomap with a c-parametrized unique fixed point has a unique 'c-parametrized fixed point'.*

Proof. By coherence and naturality, ρ gives a coalgebra morphism from δ to ν_I, so we have a 'c-parametrized fixed point' z_δ given by $z_I \circ \rho$, where z_I gives the unique fixed I-point for the endomap s gives in C_c. In a monoidal category, ρ is an isomorphism and so z_δ must be unique: a second parametrized fixed point for s would give a second fixed I-point.



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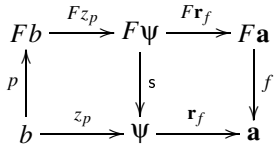
6 Fixed Point Algebras

In the rest of this paper, each category is equipped with a class of pointed objects and a class of strict maps. We use boldface variables to range over these maps and objects.

A *strict algebra morphism* is one given by a strict map **h** and a *pointed algebra* is one with a pointed carrier **a** (indicated by a dot on the structure map when the carrier is implicit). Note that a pointed algebra does not necessarily have a strict structure map and, although the terms 'strict' and 'pointed' suggest certain concrete examples, the results below hold with fairly weak conditions on the choice of pointed objects and strict maps. Note in particular that strict maps are not necessarily between pointed objects.

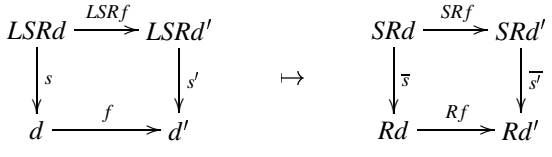
Definition 5 *A* Fixed Point Algebra *is a corecursive algebra from which there is a unique strict algebra morphism into any pointed algebra.*

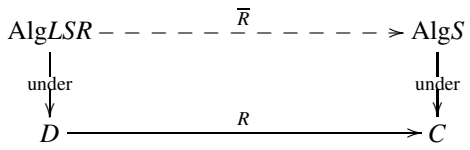


Note that Fixed Point Algebras do not necessarily have strict structure maps and are not necessarily pointed.

Theorem 2 says corecursive algebras $LSR\phi \to \phi$ transpose to corecursive algebras $SR\phi \to R\phi$. We want to extend this to Fixed Point Algebras. Whereas Theorem 2 holds with no conditions on R, L, η and S, the transposition of a Fixed Point Algebra s to a Fixed Point Algebra $\overline{\mathsf{s}}$ is less general. Given a pointed algebra s for S, we need to show that there exists a unique strict morphism $\mathbf{r}_s$ from $\overline{\mathsf{s}}$ to s. We expect to obtain this morphism from the unique strict morphism $\mathbf{r}_{\widehat{s}}$ from s to some pointed algebra $\widehat{s}$.

If s is an algebra for LSR then its transpose $\overline{s}$ is an algebra for S, without conditions on R, L, η and S. Also, if f gives an algebra morphism from s to s', then Rf gives an algebra morphism from $\overline{s}$ to $\overline{s'}$. In other words, the functor R lifts to a functor $\overline{R}$ from AlgLSR to Alg$S$.





Suppose the algebra $\widehat{s}$ is pointed and transposes to s. Suppose $\overline{R}$ preserves strictness. Taking $\mathbf{r}_s = \overline{R}\mathbf{r}_{\widehat{s}}$ then gives us a strict morphism from $\overline{\mathsf{s}}$ to s. So, if $\overline{R}$ preserves strictness and is *surjective on pointed objects*, meaning every pointed object in Alg$S$ is the image of some pointed object in AlgLSR, then $\overline{\mathsf{s}}$ is a *weak Fixed Point Algebra*, meaning the r_s exist but are not necessarily unique. Now if R preserves strictness, then so does $\overline{R}$ (which does not mean strict algebras transpose to strict algebras) and this is a natural condition to place on R. On the other hand, suppose R is surjective on pointed objects. The existence of $\mathbf{d}$ such that $\mathbf{a} = R\mathbf{d}$ does not ensure the existence of $\mathbf{d}$ such that $D(LSR\mathbf{d}, \mathbf{d})$ contains $\widehat{s}$ such that $s = \overline{R}\widehat{s}$. But suppose R is also *semi right adjoint to* L. By this we mean a natural transformation ε is given such that $f = R(\varepsilon_d \circ Lf) \circ \eta_a$ for any map f into any $a = Rd$. This is equivalent to the triangle law $R\varepsilon \circ \eta_a = \mathrm{id}_a$. Now, taking $\widehat{s}_{\mathbf{d}} = \varepsilon_{\mathbf{d}} \circ Ls$ we have $s = R\widehat{s} \circ \eta_{\mathbf{a}} = \overline{R}\widehat{s}$. Note that different choices of $\mathbf{d}$ will give different $\widehat{s}_{\mathbf{d}}$.

Proposition 6 *If R preserves strictness, is surjective on pointed objects and is semi right adjoint to L, then a Fixed Point Algebra for LSR transposes to a weak Fixed Point Algebra for S.*

To show that the $\mathbf{r}_s$ are unique requires further conditions. Given an arbitrary strict morphism $\mathbf{m}$ from $\overline{\mathsf{s}}$ to s, we need to show that $\mathbf{m} = \mathbf{r}_s$. This follows from the uniqueness of $\mathbf{r}_{\widehat{s}}$, if we assume that $\overline{R}$ is *full on strict maps* meaning every strict morphism from $\overline{R}s$ to $\overline{R}s'$ is the image of some strict morphism from s to s'. Our strict morphism $\mathbf{m}$ is then of the form $\overline{R}\widehat{\mathbf{m}}$, where $\widehat{\mathbf{m}}$ is some strict morphism from s to $\widehat{s}$, but $\mathbf{r}_{\widehat{s}}$ is the unique such morphism so $\widehat{\mathbf{m}} = \mathbf{r}_{\widehat{s}}$ and $\mathbf{m} = \overline{R}\widehat{\mathbf{m}} = \overline{R}\mathbf{r}_{\widehat{s}} = \mathbf{r}_s$.

When is $\overline{R}$ full on strict morphisms? It is not enough for R to be full on strict maps because we may have $\mathbf{m} = R\widehat{\mathbf{m}}$ without $\widehat{\mathbf{m}}$ being a morphism from s to $\widehat{s}$. Suppose, however, that R is full on strict maps and L is *semi left adjoint* to R. This means $f = \varepsilon \circ L(Rf \circ \eta)$ for all $f : Lc \to d$. Now if $\mathbf{m} = \overline{R}\widehat{\mathbf{m}}$ and $\overline{R}\widehat{\mathbf{m}}$ is a morphism from $\overline{\mathsf{s}}$ to s then $\widehat{\mathbf{m}}$ is a morphism from s to $\widehat{s}$.

Theorem 1 *If R preserves strictness, is surjective on pointed objects, full on strict maps and right adjoint to L, then a Fixed Point Algebra for LSR transposes to a Fixed Point Algebra for S.*

Applications. In applications, we find ourselves with a category D equipped with a class of pointed objects and a class of strict maps and are given a functor $R : D \to C$ along which we would like to transpose Fixed Point Algebras. What objects and maps in C should we take as pointed and strict if we would like R to satisfy the conditions of Theorem 1? If we equip C with the class of pointed objects given by applying R to pointed objects in D, then R is surjective on pointed objects by definition. If we equip C with the class of strict maps given by applying R to strict maps in D, R preserves strictness. However, despite being surjective on strict maps, R may not be full on strict maps. If $Rd_0 = Rd_1$ with $d_0 \neq d_1$, then $\mathbf{m} : d \to d_0$ and, hence, $R\mathbf{m} : Rd \to Rd_0$ may be strict without there being a strict $\mathbf{m}' : d \to d_1$ such that $R\mathbf{m}' = R\mathbf{m}$. However, if we want R to preserve strictness, we cannot force fullness on strict maps by defining $m : c \to c'$ to be strict if for all d over c and d' over c' there is a strict map $\widehat{\mathbf{m}} : d \to d'$ over m. Fortunately, this problem does not arise in the examples that interest us here.

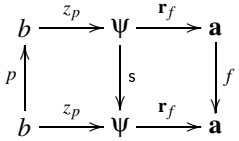
When R is the forgetful functor from domains and strict maps, when R is the forgetful functor from lift-algebras and when R is the right Kleisli adjoint for a comonad, R is injective on objects and the above definitions coincide. In the following, we therefore assume that R is injective on objects and equip C with pointed objects and strict maps as above.

1. c is pointed in C if $c = R\mathbf{d}$ for some pointed $\mathbf{d}$ in D
2. f is strict in C if $f = R\mathbf{h}$ for some strict $\mathbf{h}$ in D

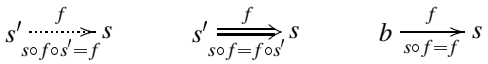
7 Fixed Point Objects

Our notion of Fixed Point Object is our notion of Fixed Point Algebra in the special case where the endofunctor is the identity. As observed in Section 4, such endomaps have unique fixed points in a particularly strong sense.

Definition 6 *A* Fixed Point Object *is a Fixed Point Algebra for the identity endofunctor.*



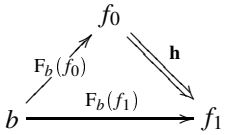
When dealing with an identity endofunctor, we must be careful to distinguish between recursive morphisms from an endomap s' to an endomap s and endomap morphisms from s' to s. However, when s' is the identity on some object b, both conditions reduce to $s \circ f = f$, which is to say f gives a fixed b-point for s.



Theorem 1, with $S = \mathrm{Id}$, relates Fixed Point Algebras for LR to Fixed Point Objects.

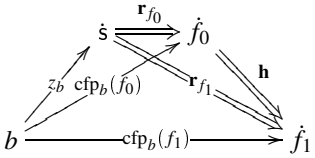
Corollary 7 *If R preserves strictness, is surjective on pointed objects, full on strict maps and right adjoint to L, then a Fixed Point Algebra for LR transposes to a Fixed Point Object.*

Each fixed b-point of a Fixed Point Object induces a well-behaved family of fixed b-points. Given an endomap f on a pointed object $\mathbf{c}$, we compose $\mathbf{r}_f$ with the unique fixed b-point z_b for s to obtain a canonical fixed b-point $\mathrm{cfp}_b(f)$ for f. A family $\mathrm{F}_b(f)$ of fixed b-points is *uniform (with respect to strict maps)* if, given any strict map $\mathbf{h}$ such that $f_1 \circ \mathbf{h} = \mathbf{h} \circ f_0$, we find that $\mathrm{F}_b(f_1) = \mathbf{h} \circ \mathrm{F}_b(f_0)$. This property is known as Plotkin's Axiom.



Proposition 7 *If strict maps are closed under composition, the family of fixed b-points induced by a Fixed Point Object is uniform.*

Proof. If strict maps compose, then strict endomap morphisms compose. So, given a strict endomap morphism $\mathbf{h}$ from f_0 to f_1, $\mathbf{h} \circ \mathbf{r}_{f_0}$ is a strict endomap morphism from s to f_1 which must then equal the unique strict endomap morphism $\mathbf{r}_{f_1}$. From $\mathbf{h} \circ \mathbf{r}_{f_0} = \mathbf{r}_{f_1}$ we have $\mathbf{h} \circ \mathbf{r}_{f_0} \circ z_b = \mathbf{r}_{f_1} \circ z_b$, which means $\mathbf{h} \circ \mathrm{cfp}_b(f_0) = \mathrm{cfp}_b(f_1)$.



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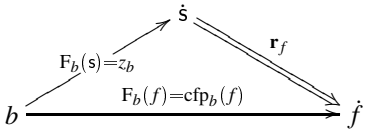
In categories of partial orders, Plotkin's Axiom characterizes least fixed points. This is a simple consequence of the fact that, when a Fixed Point Object is itself pointed, uniformity characterizes its family of fixed points.

Lemma 4 *In the presence of a* pointed *Fixed Point Object,* $\mathrm{cfp}_b(f)$ *is the unique uniform family of fixed b-points for pointed endomaps* f.

Proof. Let $\mathrm{F}_b(f)$ be a uniform family of fixed points for pointed endomaps f. Our Fixed Point Object s is pointed so this family includes a fixed b-point $\mathrm{F}_b(\mathsf{s})$ which must then equal the unique fixed b-point z_b. The endomap morphism $\mathbf{r}_f$ is strict, so uniformity gives $\mathrm{F}_b(f) = \mathbf{r}_f \circ \mathrm{F}_b(\mathsf{s})$ and hence $\mathrm{F}_b(f) = \mathbf{r}_f \circ z_b$ which means $\mathrm{F}_b(f) = \mathrm{cfp}_b(f)$.



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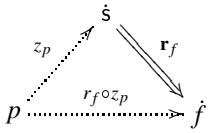
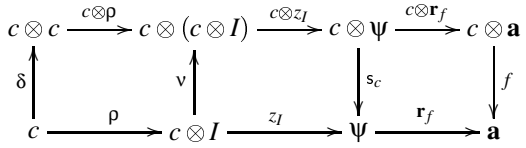
Remark. Given an endofunctor $S : C \to C$ and a Fixed Point Algebra s for S, the results above concerning fixed b-points become results about what might be called recursive p-points, where p is any coalgebra for S. For example, when s is pointed, it generates the unique family of recursive p-points for pointed algebras.

Lemma 5 *In the presence of a* pointed *Fixed Point Algebra for S,* $r_s \circ z_p$ *is the unique uniform family of recursive p-points for pointed algebras s.*



Parametrized Fixed Point Objects. As in Section 5, when S carries comonad structure, we can construct its Kleisli adjunction $K_L \dashv K_R$. The right adjoint K_R is the identity on objects and hence injective on objects. It carries a map $f : a \to b$ to the map in C_S given by $f \circ \iota_a : Sa \to b$. Strict maps in C_S are therefore maps of the form $\mathbf{h} \circ \iota$ where $\mathbf{h}$ is a strict map in C. Corollary 7, with K_L and K_R for L and R, says Fixed Point Algebras for $S = K_L \circ K_R$ transpose to Fixed Point Objects in the Kleisli category C_S *with respect to the above definition of strict map and pointed object in* C_S. In Section 5 we only needed the natural transformation $\eta : \mathrm{Id} \Rightarrow K_R K_L$, but here we are using both triangle laws of the adjunction $K_L \dashv K_R$. The one law gives the existence of r_f and, hence, the existence of a canonical family of c-parametrized fixed b-points in C_S. The other, the uniqueness of this family with respect to uniformity (assuming the Fixed Point Algebra for S is pointed).

When S is $c \otimes (\cdot)$, maps of the form $f \circ \iota_a : c \otimes a \to b$ give 'constant' maps in the Kleisli category C_c. The behavior of composition with such maps simplifies the initiality property of a Fixed Point Object in C_c. Being a strict morphism from an endomap given by $s' : c \otimes a' \to a'$ to an endomap given by $s : c \otimes a \to a$ is equivalent to being of the form $f \circ \iota_a$ with f an algebra morphism from s' to s, and so the initiality property reduces, as expected, to the algebra initiality property of the original Fixed Point Algebra. Given any map $f : c \otimes \mathbf{a} \to \mathbf{a}$, with $\mathbf{a}$ pointed, a Fixed Point Algebra s_c for $c \otimes (\cdot)$ induces both canonical fixed b-points in C_c, as described above, and a 'c-parametrized fixed point' $r_f \circ z_I \circ \rho$ (see Proposition 5).

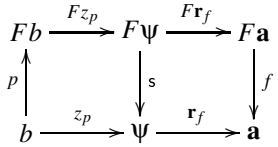


8 Freyd Algebras

The strengthened notion of free algebra introduced by Freyd in [4] is equivalent to our notion of Fixed Point Algebra in the special case where all objects are pointed and all maps strict.

Definition 7 *A* Freyd algebra *is an algebra* s *that satisfies the following equivalent conditions:*

1. s *is initial and the inverse of a final coalgebra.*
2. s *is initial and corecursive.*



Suppose all maps and objects in D are considered strict and pointed and we are given an adjunction $L \dashv R : D \to C$. We have in mind the case where R is the forgetful functor from a category of strict morphisms between pointed domains or from a category of lift algebras. Corollary 7 then says a Freyd algebra for LR transposes to a Fixed Point Object in C. Strict maps and pointed objects in C are simply those in the image of R and, assuming R is injective on objects, the image of R forms a subcategory of C. In particular, strict maps in C are closed under composition and our Fixed Point Object induces a uniform family of fixed points by Proposition 7. Moreover, because the original Freyd Algebra is pointed (all objects are pointed in D) and transposition preserves pointedness, our Fixed Point Object is pointed and hence, by Lemma 4, induces the unique uniform family of fixed points in C (with respect to strict morphisms between endomaps on pointed objects).

If we are also given an endofunctor S on C, Theorem 1 then says a Freyd algebra for LSR transposes to Fixed Point Algebra for S. If S carries comonad structure, we obtain a Fixed Point Object in C_S as described in the previous section or we can transpose the Freyd algebra for LSR directly to a Fixed Point Object in C_S using Corollary 7 with $K_R \circ R$ and $L \circ K_L$ for L and R. Either way, strict maps in C_S are closed under composition so we obtain a canonical family of parametrized fixed points and our Fixed Point Object is pointed so this is the unique such family.

We should mention another definition that also generates well-behaved families of fixed points and which is, *a priori*, more general than the notion of Fixed Point Algebra.

Definition 8 *A* Mulry algebra *is an initial algebra that transposes to a corecursive algebra.*

In Mulry's setting [5], which has an endofunctor $T : C \to C$ together with a natural transformation $\eta : \mathrm{Id} \Rightarrow T$, this means an initial algebra $\sigma : T\phi \to \phi$ such that $\sigma \circ \eta$ is corecursive, and therefore has a unique fixed point. In our setting, which has a natural transformation $\eta : \mathrm{Id} \Rightarrow RL$, this means an initial algebra $\sigma : LSR\phi \to \phi$ such that $R\sigma \circ \eta$ is corecursive. When $T = RL$, the two interpretations are equivalent. If σ is initial for LR, then $R\sigma$ is initial for RL and any initial algebra for RL is isomorphic to one in this form. Corecursive algebras transpose to corecursive algebras, so Freyd algebras are Mulry algebras. This is all true whether or not T is a monad or $L \dashv R$. If η *is* the unit of a monad or adjunction the notion of Mulry Fixed Point Algebra is equivalent to our notion of Fixed Point Object in C if we take 'strict' maps and 'pointed' objects to mean those in the image of T or R.

LRLR. By Freyd's Iterated Square, a Freyd algebra for $LRLR$ is of the form $\sigma \circ LR\sigma$, where σ is a Freyd algebra for LR, and, by the dinaturality of Freyd algebra delivery, $R\sigma$ is a Freyd algebra for RL, a Fixed Point Algebra with all maps and objects strict and pointed. On the other hand, Theorem 1, with $S = RL$, says a Freyd algebra for $LRLR$ transposes to a Fixed Point Algebra for the *monad* RL, where we are assuming strict maps and pointed objects are given by the image of R. We don't know what to make of this second algebra for RL.

Acknowledgement

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